TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 280, Number 2, December 1983

UNIQUENESS OF TORSION FREE CONNECTION ON SOME INVARIANT STRUCTURES ON LIE GROUPS

BY

MICHEL NGUIFFO BOYOM AND GEORGES GIRAUD

ABSTRACT. Let $\mathcal{G}$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{Int}(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$. The group $\mathcal{G}$ is naturally equipped with $\mathrm{Int}(\mathfrak{g})$-reductions of the bundle of linear frames on $\mathcal{G}$. We investigate for what kind of Lie group the 0-connection of E. Cartan is the unique torsion free connection adapted to any of those $\mathrm{Int}(\mathfrak{g})$-reductions.

1. Definitions and main results. Let M be an n-dimensional manifold and let G be a Lie subgroup of the linear group $Gl(\mathbf{R}^n)$, with Lie algebra $\mathfrak{G}$. All manifolds we shall consider are smooth and connected. Let us consider a G-reduction $E(M, G)$ of the frame bundle $E^0(M, Gl(\mathbf{R}^n))$ and two linear connections ∇_1 and ∇_2 adapted to $E(M, G)$. Suppose these connections have the same torsion tensor, so that

$$(\nabla_1)_X Y - (\nabla_1)_Y X - [X, Y] = (\nabla_2)_X Y - (\nabla_2)_Y X - [X, Y]$$

or

$$(\nabla_1 - \nabla_2)_X Y - (\nabla_1 - \nabla_2)_Y X = 0$$

for any vector fields X, Y on M. Then if one identifies the tangent space $T_x(M)$ for $x \in M$, with $\mathbf{R}^n$, the difference $\nabla_1 - \nabla_2$ appears as an element of the space $\mathbf{R}^{n*} \otimes \mathfrak{G} \cap S^2\mathbf{R}^{n*} \otimes \mathbf{R}^n$ which is known to be the first prolongation of $\mathfrak{G}$ (see **[4]**).

A G-structure $E(M, G)$ is said to be 1-flat if it can be equipped with a torsion free linear connection. Thus any 1-flat G-structure can be equipped with at most one torsion free linear connection if and only if the first prolongation of $\mathfrak{G}$ is zero.

We are concerned with the following problem. Let (M, ω) be a differentiable manifold equipped with a torsion free linear connection ω. We wish to describe those linear subgroups G such that the connection ω is the unique linear connection adapted to some G-reduction of the frame bundle of M. Obviously a necessary condition is that the first prolongation of the holonomy algebra of ω be zero. So if $\mathcal{H}_\omega$ is the holonomy algebra of ω, the problem of finding all linear Lie groups with the previous properties is equivalent to that of finding all Lie subalgebras $\mathfrak{G}$ of $\mathrm{End}(\mathbf{R}^n)$ such that

(p_1) $$\mathcal{H}_\omega \subset \mathfrak{G},$$

(p_2) $$\mathfrak{G}^{(1)} = 0.$$

Received by the editors October 19, 1981 and, in revised form, July 27, 1982 and January 3, 1983.

1980 *Mathematics Subject Classification*. Primary 53C10, 53C05; Secondary 22E60.

Key words and phrases. G-structure, torsion free connection, prolongation, symmetric operator of Lie algebra.

In what follows we deal with differentiable manifolds $(\mathcal{G}, \omega)$ which are Lie groups equipped with the 0-connection of Cartan. Our interest in this particular case is due to the fact that the 0-connection of Cartan describes the local structure of the Lie group $\mathcal{G}$. In particular, the 0-connection of Cartan is invariant by the left translations of $\mathcal{G}$, while if ∇ is the covariant derivative associated to the 0-connection then $\nabla_X Y = \frac{1}{2}[X, Y]$ for any two left invariant vector fields X and Y on $\mathcal{G}$.

As a direct consequence of the above facts one deduces that the holonomy algebra, $\mathcal{H}_0$, of the 0-connection is included in the Lie algebra $\mathrm{ad}(\mathfrak{g})$ of inner derivations of $\mathfrak{g}$ ($\mathfrak{g}$ being the Lie algebra of $\mathcal{G}$). Let $\mathrm{Int}(\mathfrak{g})$ be the connected Lie subgroup of $Gl(\mathfrak{g})$ associated to $\mathrm{ad}(\mathfrak{g})$. Let us extend the holonomy fiber bundles of the 0-connection to $\mathrm{Int}(\mathfrak{g})$-reductions of the frame bundle of $\mathcal{G}$ to get left invariant $\mathrm{Int}(\mathfrak{g})$-structures. Any two such extensions are conjugate.

Our main results give a characterization of those Lie groups $\mathcal{G}$ on which the $\mathrm{Int}(\mathfrak{g})$-structures constructed as above belong to the set of $\mathrm{Int}(\mathfrak{g})$-reductions of the frame bundle of $\mathcal{G}$ which satisfy the properties (p_1) and (p_2), so that $\mathcal{H}_0 \subset \mathrm{ad}(\mathfrak{g})$ and $(\mathrm{ad}(\mathfrak{g}))^{(1)} = 0$. For such a Lie group $\mathcal{G}$, the 0-connection of Cartan is the unique torsion free linear connection adapted to its holonomy bundles. We make technical use of a Lie subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of the linear Lie algebra $\mathrm{End}(\mathfrak{g})$, which is defined as follows. A linear endomorphism φ of the vector space $\mathfrak{g}$ belongs to $\mathfrak{h}_{\mathfrak{g}}$ if it satisfies the identity

$$[\varphi(X), Y] + [X, \varphi(Y)] = 0$$

for any pair (X, Y) in $\mathfrak{g} \times \mathfrak{g}$. Such a φ is called a symmetric operator of $\mathfrak{g}$. In the present work we restrict ourselves to the case of nonsolvable Lie groups.

Now let us denote by $\mathfrak{r}$ the radical of the Lie algebra $\mathfrak{g}$, i.e., $\mathfrak{r}$ is the maximal solvable ideal in $\mathfrak{g}$. Taking a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, the vector space $\mathfrak{g}$ becomes a direct sum: $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. Let us consider $\mathfrak{r}$ with its $\mathfrak{s}$-module structure given by the extension $0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{s} \to 0$. The subspace of $\mathfrak{r}$ consisting of $\mathfrak{s}$-invariant elements is denoted by $\mathfrak{r}^{\mathfrak{s}}$. As $\mathfrak{s}$ is a semisimple Lie algebra, the subspace $[\mathfrak{s}, \mathfrak{r}]$ is a submodule of the $\mathfrak{s}$-module $\mathfrak{r}$, and one gets the direct sum of $\mathfrak{s}$-modules

$$\mathfrak{r} = \mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}].$$

The maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{r}^{\mathfrak{s}}$ is denoted $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ and the center of the subalgebra $\mathfrak{r}^{\mathfrak{s}}$ is denoted $Z(\mathfrak{r}^{\mathfrak{s}})$. The subspace of $\mathfrak{r}^{\mathfrak{s}}$ denoted by $h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}})$ is that obtained by the evaluation map of $h_{\mathfrak{r}^{\mathfrak{s}}} \otimes \mathfrak{r}^{\mathfrak{s}}$ in $\mathfrak{r}^{\mathfrak{s}}$.

The main geometrical results to be proved are the following.

$(\mathfrak{R}_1)$ Let $\mathcal{G}$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Then the 0-connection ∇_0 of Cartan is the unique torsion free connection on each $\mathrm{Int}(\mathfrak{g})$-extension of any holonomy bundle of ∇_0 if and only if the ideal $h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}}) \cap D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is included in the center of $\mathfrak{r}^{\mathfrak{s}}$.

$(\mathfrak{R}_2)$ Let $\mathcal{G}$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Suppose that $\mathfrak{r}^{\mathfrak{s}}$ is a commutative subalgebra of $\mathfrak{g}$. Then one gets uniqueness of torsion free connection adapted to each $\mathrm{Int}(\mathfrak{g})$-extension of any holonomy bundle of ∇_0 if and only if the Lie group $\mathcal{G}$ has discrete center.

($\mathcal{R}_3$) Take $\mathcal{G}$ to be a Lie group, the radical of which is nilpotent subgroup. If $\mathcal{G}$ has discrete center, then there is a unique torsion free connection on each Int($\mathfrak{g}$)-extension of the holonomy bundle of ∇_0.

($\mathcal{R}_4$) Given a Lie group $\mathcal{G}$, let $\mathcal{R}^{\mathfrak{s}}$ be the connected Lie subgroup of $\mathcal{G}$ associated to the Lie subalgebra $\mathfrak{r}^{\mathfrak{s}}$. If $\mathcal{R}^{\mathfrak{s}}$ is a normal subgroup, then one gets uniqueness of the torsion free connection on Int($\mathfrak{g}$)-extension of the holonomy bundle of ∇_0 if and only if the same result holds on the Lie group $\mathcal{R}^{\mathfrak{s}}$.

2. Algebraic results. Because of the left invariant character of the previous results we shall deal with their infinitesimal versions. Thus, at the Lie algebra level we are concerned with finite-dimensional Lie algebras on a field K of characteristic zero.

THEOREM 1. *For any linear endomorphism φ of $\mathfrak{g}$ which belongs to the Lie algebra $h_{\mathfrak{g}}$ the following assertions hold*:

(i) *The restriction of φ to the subspace $[\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$ takes its values in the center of the Lie algebra $\mathfrak{g}$.*

(ii) *The restriction of φ to the Lie algebra $\mathfrak{r}^{\mathfrak{s}}$ is an element of the Lie algebra $h_{\mathfrak{r}^{\mathfrak{s}}}$ and takes its values in the subspace $\mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$ of elements in $\mathfrak{r}$ which commute with the subspace $[\mathfrak{r}, \mathfrak{s}]$.*

Before starting the proof of Theorem I, let us remark that our interest in the Lie algebra $h_{\mathfrak{g}}$ arises from the following facts. Let $\mathfrak{g}$ be a Lie algebra and let $h^0_{\mathfrak{g}}$ be the vector space of all linear maps of $\mathfrak{g}$ into its center $Z(\mathfrak{g})$. Consider the linear map π of $h_{\mathfrak{g}}$ into $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ given by $\pi(\varphi) = \mathrm{ad} \circ \varphi$, $\varphi \in h_{\mathfrak{g}}$, so that for any element X in $\mathfrak{g}$ one gets $\pi(\varphi)(X) = \mathrm{ad}_{\varphi(X)}$. It is clear that the bilinear map $(X, Y) \to [\varphi(X), Y]$ of $\mathfrak{g} \times \mathfrak{g}$ in $\mathfrak{g}$ is symmetric. Thus the previous map π takes its values in the first prolongation of ad($\mathfrak{g}$). This map is onto because of the definition of $(\mathrm{ad}(\mathfrak{g}))^{(1)}$. The kernel of π is $h^0_{\mathfrak{g}}$. So one obtains the following exact sequence of vector spaces:

$$0 \to h^0_{\mathfrak{g}} \to h_{\mathfrak{g}} \to (\mathrm{ad}(\mathfrak{g}))^{(1)} \to 0.$$

Since the Cartan-Killing form $(X, Y) \mapsto \Phi(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ of $\mathfrak{g}$ is invariant by the inner derivations of $\mathfrak{g}$, for each φ in $h_{\mathfrak{g}}$ one gets

$$\begin{aligned}\Phi([\varphi(X), Y], Z) &= -\Phi(Y, [\varphi(X), Z]) = -\Phi(Y, [\varphi(Z), X]) \\ &= \Phi([\varphi(Z), Y], X) = \Phi([\varphi(Y), Z], X) = -\Phi(Z, [\varphi(Y), X]) \\ &= -\Phi([\varphi(Y), X], Z) = -\Phi([\varphi(X), Y], Z).\end{aligned}$$

Thus $\Phi([\varphi(X), Y], Z) = \Phi(\varphi(X), [Y, Z]) = 0$, and the image $\varphi(\mathfrak{g})$ is perpendicular to $[\mathfrak{g}, \mathfrak{g}]$ under Φ. As is well known this implies that $\varphi(\mathfrak{g})$ lies in the radical $\mathfrak{r}$ of $\mathfrak{g}$.

For an element φ in $h_{\mathfrak{g}}$ let us denote by A and B the restriction of φ to $\mathfrak{r}$ and to $\mathfrak{s}$, respectively. Let (r, s) and (r', s') be two elements in $\mathfrak{g} \approx \mathfrak{r} \times \mathfrak{s}$. With respect to above notation one gets

$$[(A(r) + B(s), 0), (r', s')] = [(A(r') + B(s'), 0), (r, s)].$$

This last identity gives rise to the system

$$[A(r), r'] = [A(r'), r], \tag{1}$$

(2) $$[B(s), r'] = [A(r'), s],$$

(3) $$[B(s), s'] = [B(s'), s].$$

To prove Theorem I, we need two technical lemmas.

LEMMA 1. *Let* $\mathfrak{g}$ *be a Lie algebra such that its Levi subalgebras* $\mathfrak{s}$ *are* 3-*dimensional, and let* $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ *be a Levi decomposition of* $\mathfrak{g}$. *Suppose* $\mathfrak{r}$ *is an irreducible* $\mathfrak{s}$-*module of dimension greater than one. Then for any element* φ *of* $h_{\mathfrak{g}}$ *the restriction* B *of* $\mathfrak{s}$ *to* φ *is zero.*

PROOF. One can suppose the ground field is algebraically closed. (This is done without loss of generality.) Let $m + 1$ be the dimension of the radical of $\mathfrak{g}$. Since $\mathfrak{s}$ is a 3-dimensional semisimple Lie algebra, we can choose a basis (X, Y, H) in $\mathfrak{s}$ such that

(4) $$[X, Y] = H, \qquad [H, X] = 2X, \qquad [H, Y] = -2Y.$$

Let v be a primitive element of the $\mathfrak{s}$-module $\mathfrak{r}$. Then the sequence $v_0 = v$, $v_1 = Yv, \ldots, v_m = Y^m v$ is a basis of the vector space $\mathfrak{r}$ which satisfies the system

(5) $$\begin{gathered} H.v_i = (m - 2i)v_i, \qquad i = 0, 1, \ldots, m, \\ Y.v_i = v_{i+1}, \qquad i = 0, 1, \ldots, m-1 \text{ and } Y.v_m = 0, \\ X.v_0 = 0 \quad \text{and} \quad X.v_i = (-mi + i(i-1))v_{i-1}, \qquad i = 1, \ldots, m, \end{gathered}$$

where, for any $s \in \mathfrak{s}$ and $r \in \mathfrak{r}$ we write $s.r$ for $[s, r]$. Now from the relations (3) and the system (4) one obtains

$$Y.B(H) = H.B(Y), \qquad Y.B(X) = X.B(Y), \qquad H.B(X) = X.B(H).$$

If one writes these in terms of the basis (v_i), one gets

(6) $$\begin{gathered} \sum_{i=0}^{m} B_i(H) Y.v_i = \sum_{i=0}^{m} B_i(Y) H.v_i, \\ \sum_{i=0}^{m} B_i(X) Y.v_i = \sum_{i=0}^{m} B_i(Y) X.v_i, \\ \sum_{i=0}^{m} B_i(X) H.v_i = \sum_{i=0}^{m} B_i(H) X.v_i, \end{gathered}$$

the v_i-components in (6) for $i = 0, 1, \ldots, m$, we have the relations

$$B_0(Y) = 0, \qquad B_1(Y) = 0, \qquad B_{m-1}(X) = 0, \qquad B_m(X) = 0,$$

and for $1 \leqslant i \leqslant m - 1$,

$$\begin{gathered} B_{i-1}(H) = (m - 2i) B_i(Y), \\ B_{i-1}(X) = (i + 1)(-m + i) B_{i+1}(Y), \\ (m - 2i) B_i(X) = (i + 1)(-m + i) B_{i+1}(Y). \end{gathered}$$

The last three equalities give

$$(i + 2)(m - 2i)(-m + i + 1) B_{i+2}(Y) = (i + 1)(-m + i)(m - 2i - 4) B_{i+2}(Y).$$

Therefore, we get either $B_{i+2}(Y) = 0$ or

$$(i+2)(m-2i)(m-i-1) = (i+1)(m-i)(m-2i-4).$$

The ultimate equality implies $m(m+2) = 0$; that cannot hold because m is positive. If i is an integer such that $2 < i+2 < m$ one gets $B_{i+2}(Y) = 0$. This proves that $B(Y) = 0$ and we conclude that $B(X) = B(H) = 0$. Now we show that we can drop the condition that $\mathfrak{r}$ is an irreducible $\mathfrak{s}$-module.

LEMMA 2. *Let* $\mathfrak{g}$ *be a Lie algebra such that its Levi subalgebras* $\mathfrak{s}$ *are* 3-*dimensional and let* $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ *be a Levi decomposition of* $\mathfrak{g}$. *For any element* φ *of* $h_{\mathfrak{g}}$ *the restriction* B *of* φ *to* $\mathfrak{s}$ *takes its values in the center* $Z(\mathfrak{g})$.

PROOF. As in Lemma 1, let us suppose that the ground field is algebraically closed. Because of the simplicity of $\mathfrak{s}$, the radical $\mathfrak{r}$ is a direct sum of irreducible $\mathfrak{s}$-modules

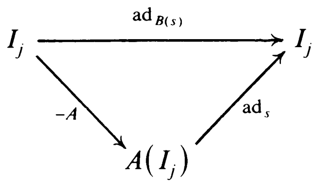
$$\mathfrak{r} = I_1 \oplus \cdots \oplus I_t. \tag{7}$$

We know that if φ is an element in $h_{\mathfrak{g}}$, the linear map B of $\mathfrak{s}$ into $\mathfrak{r}$ which is deduced from φ satisfies the relation $[B(s), s'] = [B(s'), s]$. Take B_{I_j} to be the I_j-component of B. Then Lemma 1 tells us that, for any I_j which has dimension greater than one, we get $B_{I_j} = 0$, so that B takes its values in the subalgebra $\mathfrak{r}^{\mathfrak{s}}$. The relation $[B(s), r] = [A(r), s]$ implies that $[B(s), r]$ lies in the $\mathfrak{s}$-module $[\mathfrak{s}, \mathfrak{r}]$. The subspace $\mathfrak{r}^{\mathfrak{s}}$ being a subalgebra of $\mathfrak{g}$, the term $[B(s), r_0]$ also lies in $\mathfrak{r}^{\mathfrak{s}}$ for any (s, r_0) in $\mathfrak{s} \times \mathfrak{r}^{\mathfrak{s}}$, so that we get $[B(s), r_0] = 0$. Therefore, we see that B takes its values in the center of $\mathfrak{r}^{\mathfrak{s}}$. Thus if s and s' are elements of $\mathfrak{s}$ and if $r \in \mathfrak{r}$ we get

$$[s', [B(s), r]] = [[s', B(s)], r] + [B(s), [s', r]] = [B(s), [s', r]],$$

so that the inner derivation $\mathrm{ad}_{B(s)}$ of $\mathfrak{r}$ is compatible with the action of $\mathfrak{s}$. This means that $\mathrm{ad}_{B(s)}$ is a $\mathfrak{s}$-module morphism.

Suppose that I_j is an irreducible factor of the decomposition (7) with $\dim I_j > 1$. The classical Schur lemma tells us that either $\mathrm{ad}_{B(s)}(I_j)$ is $\{0\}$ or $\mathrm{ad}_{B(s)|I_j}$ is an isomorphism. In the latter case the subspace $J_j = [B(s), I_j]$ is an irreducible $\mathfrak{s}$-module which is not zero. According to the formula (2), one gets the following commutative diagram:



If J_j is different from $\{0\}$ the above diagram implies that A is an isomorphism of I_j on $A(I_j)$ and idem for the restriction to $A(I_j)$ of ad_s. We conclude that $A(I_j)$ is exactly the submodule J_j. Moreover, J_j does not depend to the choice of s in $\mathfrak{s}$. As we deal only with restrictions, the map $s \to \mathrm{ad}_{B(s)} | I_j$ is a linear map of $\mathfrak{s}$ in the space of $\mathfrak{s}$-morphisms of I_j into J_j, so that the Schur lemma implies that the above map has rank one. Finally, we deduce from the relation $-\mathrm{ad}_{B(s)|I_j} \circ A^{-1}_{|I_j} = \mathrm{ad}_s | I_j$ that the kernel of $s \to \mathrm{ad}_{B(s)}|I_j$ is a nonzero ideal of $\mathfrak{s}$ different from $\mathfrak{s}$. The Lie algebra $\mathfrak{s}$ being simple, we get a contradiction.

PROOF OF THEOREM I. Let us keep in mind that the ground field is algebraically closed. Let C be a Cartan subalgebra of the Lie algebra $\mathfrak{s}$. Fix a simple system of roots $\mathcal{Q} = (\alpha_1, \dots, \alpha_k)$ associated to C. We write (X_i, Y_i, H_i) for the Weyl system $(X_{\alpha_i}, Y_{\alpha_i}, H_{\alpha_i})$ corresponding to the system $\mathcal{Q}$. As vector space, the Lie algebra $\mathfrak{s}$ is generated by the system (X_i, Y_i, H_i), $i = 1, 2, \dots, k$. Let (n_{ij}), $i, j = 1, 2, \dots, k$, be the Cartan matrix which is associated to $\mathcal{Q}$. For any $i = 1, 2, \dots, k$, let $\mathfrak{s}_i$ be the Lie algebra $KX_i \oplus KY_i \oplus KH_i$ (see [**2**, Chapter IV, §3]). Now let φ be an element of $h_{\mathfrak{g}}$. Lemma 2 guarantees that the vector subspace $\varphi(\mathfrak{s}_i) = B(\mathfrak{s}_i)$ is contained in the center of the subalgebra $\mathfrak{r} \oplus \mathfrak{s}_i$ of $\mathfrak{g}$. If we consider the 2-cochain $X, Y \to -B[X, Y]$, then (3) is equivalent to the fact that the above 2-cochain is the coboundary of the 1-cochain $X \to B(X)$. Thus the 2-cochain $X, Y \to -B[X, Y]$ must be closed, so that

$$(8) \qquad [X, B[Y, Z]] - [Y, B[X, Z]] + [Z, B[X, Y]] = 0$$

for any (X, Y, Z) in $\mathfrak{s} \times \mathfrak{s} \times \mathfrak{s}$. Now take i, j in $[1, 2, \dots, k]$. According to the Weyl relations we may deduce from (8) that $[X_i, B[H_i, Y_j]] - [Y_j, B[H_i, X_i]] = -n_{ij}[X_i, B(Y_j)] - 2[Y_j, B(X_i)] = -(n_{ij} + 2)[X_i, B(Y_j)] = 0$. On the other hand, the relation (3) gives

$$(n_{ij} + 2)[X_i, B(Y_j)] = (n_{ji} + 2)[X_i, B(Y_i)] = 0.$$

For these last equalities to hold, the necessary condition is

$$(9) \qquad [X_i, B(Y_j)] = 0.$$

Now let us compute the quantity $[H_i, B(Y_j)]$, taking

$$\begin{aligned}[H_i, B(Y_j)] &= [[X_i, Y_i], B(Y_j)] = [[X_i, B(Y_j)], Y_i] + [X_i, [Y_i, B(Y_j)]] \\ &= [X_i, [Y_i, B(Y_j)]] = [X_i, [Y_j, B(Y_i)]] \\ &= [[X_i, Y_j], B(Y_i)] + [Y_j, [X_i, B(Y_i)]].\end{aligned}$$

If $i \neq j$, the Weyl relations together with Lemma 2 give

$$[[X_i, Y_j], B(X_i)] = 0 \quad \text{and} \quad [Y_j, [X_i, BY_i]] = 0,$$

so that, for any i, j in $[1, 2, \dots, k]$,

$$(10) \qquad [H_i, B(Y_j)] = 0.$$

Finally, (9) and (10) tell us that for any $j = 1, 2, \dots, k$ the element $\varphi(Y_j) = B(Y_j)$ (when it is not zero) is a primitive element in the $\mathfrak{s}$-module $\mathfrak{r}$ with the weight $0 \in C^*$. Therefore let us denote by $\mathfrak{m}_j$ the irreducible $\mathfrak{s}$-module generated by $\varphi(Y_j)$. It is well known that $\mathfrak{m}_j$ is generated as a vector space by the system $Y_1^{m_1} Y_2^{m_2} \cdots Y_k^{m_k}.B(Y_j)$ where one identifies Y_i with the operator $B(Y_j) \to [Y_i, B(Y_j)]$. On the other hand,

$Y_1^{m_1} \cdots Y_k^{m_k}.B(Y_j)$ has the weight $-\Sigma_{i=1}^k m_i\alpha_i$. In particular, let us compute the quantity $[H_t,[Y_i, BY_j]]$, taking

$$\begin{aligned}
[H_t,[Y_i, B(Y_j)]] &= [[X_t, Y_t],[Y_i, B(Y_j)]] \\
&= [X_t,[Y_t,[Y_i, B(Y_j)]]] - [Y_t,[X_t,[Y_i, B(Y_j)]]] \\
&= [X_t,[Y_t,[Y_i, B(Y_j)]]] = [X_t,[[Y_t, Y_i], B(Y_j)]] + [X_t,[Y_i,[Y_t, B(Y_j)]]] \\
&= [X_t,[[Y_t, Y_i], B(Y_j)]] = [X_t,[Y_j, B([Y_t, Y_i])]] = [Y_j,[X_t, B([Y_t, Y_i])]] \\
&= [Y_j,[[Y_t, Y_i], B(X_t)]] = [Y_j,[[Y_t, B(X_t)], Y_i]] + [Y_j,[Y_t,[Y_i, B(X_t)]]] \\
&= [Y_j,[Y_t,[Y_i, B(X_t)]]] = [Y_j,[Y_t,[X_t, B(Y_i)]]] = 0.
\end{aligned}$$

This gives the identity

$$[H_t,[Y_i, BY_j]] = 0 \tag{11}$$

for any i, j, t in $[1,2,\dots,k]$. From the formulas (9) and (10) one gets $[X_t,[Y_i, B(Y_j)]] = 0$ for any i, j, t in $[1,2,\dots,k]$. Thus (11) implies that $[Y_i, B(Y_j)]$ (if not zero) is a primitive element in $\mathfrak{r}$ with weight $0 \in K^*$. This contradicts the fact that any $Y_iB(Y_j) = [Y_i, B(Y_j)]$ is associated to the weight $-\alpha_i$. We see that B takes its values in the center of the Lie algebra $\mathfrak{g}$, which proves part of (i). Let A_0 (resp. A_1) be the restriction to $\mathfrak{r}^{\mathfrak{s}}$ (resp. to $[\mathfrak{r}, \mathfrak{s}]$) of $\varphi \in h_{\mathfrak{g}}$. It is a consequence of the exact sequence $0 \to h_{\mathfrak{g}}^0 \to h_{\mathfrak{g}} \to \mathrm{ad}(\mathfrak{g})^{(1)} \to 0$ that the subspace $h_{\mathfrak{g}}(\mathfrak{g})$ generated by all the $\varphi(X)$, $\varphi \in h_{\mathfrak{g}}$, $X \in \mathfrak{g}$, is an ideal of the Lie algebra $\mathfrak{g}$. In fact, take (X, φ) in $\mathfrak{g} \times h_{\mathfrak{g}}$ and define $X\varphi$ to be the element of $\mathrm{End}(\mathfrak{g})$ defined by $Y \to (X\varphi)(Y) = [X, \varphi(Y)] - \varphi[X, Y]$. One easily verifies that the map $X\varphi$ belongs to $h_{\mathfrak{g}}$, so that for any X and X' in $\mathfrak{g}$ and for any φ in $h_{\mathfrak{g}}$ the element $[X, \varphi(X')]$ lies in $h_{\mathfrak{g}}(\mathfrak{g})$. Now take r and r' in $\mathfrak{r} = \mathfrak{r}^{\mathfrak{s}} + [r, s]$. We may write

$$r = r_0 + r_1, \qquad r' = r_0' + r_1'$$

where r_0 and r_0' (resp. r_1 and r_1') belong to $\mathfrak{r}^{\mathfrak{s}}$ (resp. to $[\mathfrak{r}, \mathfrak{s}]$), to get

$$[A_0r_0 + A_1r_1, r_0' + r_1'] = [A_0r_0' + A_1r_1', r_0 + r_1].$$

This equation yields the three identities

$$[A_1r_1, r_1'] = [A_1r_1', r_1], \tag{12}$$

$$[A_0r_0, r_0'] = [A_0r_0', r_0], \tag{13}$$

$$[A_0r_0, r_1'] = [A_1r_1', r_0]. \tag{14}$$

Given an element s in $\mathfrak{s}$, (14) implies

$$[s,[A_1r_1, r_1']] = [[s, A_1r_1], r_1'] + [A_1r_1,[s, r_1']].$$

Relation (2) together with Lemma 2 implies that the ideal $h_{\mathfrak{g}}(\mathfrak{g})$ lies in the subalgebra $\mathfrak{r}^{\mathfrak{s}}$, so that we get

$$[s,[A_1r_1, r_1']] = [A_1r_1,[s, r_1']].$$

The first member $[s,[A_1r_1, r_1']]$ lies in the subspace $[\mathfrak{r}, \mathfrak{s}]$, while the second member lies in the ideal $h_{\mathfrak{g}}(\mathfrak{g})$, so that $[A_1r_1,[s, r_1']] = 0$, and we obtain the equality

$$[A_1r_1,[\mathfrak{r}, \mathfrak{s}]] = \{0\}.$$

Bracketing $s \in \mathfrak{s}$ with both sides of (14) one gets

$$[s,[A_0r_0, r_1']] = [A_0r_0,[s, r_1']] = [s,[A_1r_1', r_0]] = 0.$$

Our conclusion is

$$[A_0r_0,[s, r_1']] = [A_1[s, r_1'], r_0] = 0.$$

That ends the proof of (i). Proving (i), we established (13) and $[A_0r_0,[\mathfrak{r}, \mathfrak{s}]] = \{0\}$, so that (ii) holds and Theorem I is proved.

Applying Theorem I to a particular situation, we get the following

COROLLARY I.1. *Let $\mathfrak{g}$ be a Lie algebra. Keeping the previous notations, suppose that the subalgebra $\mathfrak{r}^{\mathfrak{s}}$ is commutative. Then the Lie algebra $h_{\mathfrak{g}}$ is zero if and only if the center of $\mathfrak{g}$ is zero.*

PROOF. First, suppose that $h_{\mathfrak{g}}$ is zero. Then because of the inclusion of $h_{\mathfrak{g}}^0 = Z(\mathfrak{g}) \otimes \mathfrak{g}^*$ in $h_{\mathfrak{g}}$ the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is zero. Second, suppose the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is zero. Let φ be an element of $h_{\mathfrak{g}}$. For any element r_0 in $\mathfrak{r}^{\mathfrak{s}}$ the assertion (ii) of Theorem I tells us that the element $\varphi(r_0)$ commutes with the subspace $[\mathfrak{s}, \mathfrak{r}]$. Since $\mathfrak{r}^{\mathfrak{s}}$ is supposed to be commutative, $\varphi(r_0)$ lies in the center of $\mathfrak{g}$, which implies that the map φ is identically zero.

EXAMPLE 2.1. Let $\mathcal{G}$ be any semisimple connected Lie group with Lie algebra $\mathfrak{g}$. Theorem I tells us that the Lie algebra $h_{\mathfrak{g}}$ is zero, so that the exact sequence $0 \to h_{\mathfrak{g}}^0 \to h_{\mathfrak{g}} \to (\mathrm{ad}(\mathfrak{g}))^{(1)} \to 0$ gives $\mathrm{ad}(\mathfrak{g})^{(1)} = \{0\}$.

Keeping our previous notations, we have the following result.

THEOREM II. *Let $\mathfrak{g}$ be a Lie algebra and let us denote by $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{r}^{\mathfrak{s}}$. For any decomposition $\mathfrak{g} = \mathfrak{r} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$ we have* $h_{\mathfrak{g}} = \mathrm{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus (h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})))$.

PROOF OF THEOREM II. Let us recall the construction of $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ as it is given in [**1**]. We define the sequence $D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})$ by setting $D_{\mathfrak{g}}^{0}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{r}^{\mathfrak{s}}$ and $D_{\mathfrak{g}}^{i+1}(\mathfrak{r}^{\mathfrak{s}}) = D^1(D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})) = \{X \in D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})/[X, \mathfrak{g}] \subset D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})\}$, $i \geqslant 0$. The ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is the limit of the sequence $D_{\mathfrak{g}}^{i}(\mathfrak{r}^{\mathfrak{s}})$.

First let us observe that $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is equal to $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$. Indeed, if (r_0, r_1, s) is an element of $\mathfrak{r}^{\mathfrak{s}} \times \mathfrak{r} \times \mathfrak{s}$, we have $[s,[r_0, r_1]] = [r_0,[s, r_1]]$. The first member $[s,[r_0, r_1]]$ lies in the subspace $[\mathfrak{r}, \mathfrak{s}]$, so that we have the inclusion $[\mathfrak{r}^{\mathfrak{s}},[\mathfrak{r}, \mathfrak{s}]] \subset [\mathfrak{r}, \mathfrak{s}]$. Now, if X is an element of $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$ and $(r_0, r_1) \in \mathfrak{r}^{\mathfrak{s}} \times [\mathfrak{r}, \mathfrak{s}]$, we get

$$[[X, r_0], r_1] = [[X, r_1], r_0] + [X,[r_0, r_1]] = 0.$$

We conclude that $\mathrm{ad}_X(\mathfrak{g})$ is included in $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$, so that $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$ is included in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$. Conversely, let (x, y, s) be an element of $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) \times \mathfrak{r} \times \mathfrak{s}$, so we have $[[x, y], s] - [x,[y, s]] = 0$. The term $[[x, y], s]$ belongs to $[\mathfrak{r}, \mathfrak{s}]$ while $[x,[y, s]]$

belongs to $[\mathfrak{r}, \mathfrak{s}]$, so that $\operatorname{ad}_x([\mathfrak{r}, \mathfrak{s}]) = \{0\}$. Once we get $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$, Theorem I implies the inclusion

$$h_{\mathfrak{g}} \subset \operatorname{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus h_{\mathfrak{r}^{\mathfrak{s}}} \cap \operatorname{Hom}_K\big(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})\big).$$

Conversely, any element (φ_0, φ_1) of $\operatorname{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus h_{\mathfrak{r}^{\mathfrak{s}}} \cap \operatorname{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}))$ defines a unique element $\hat{\varphi}$ of $h_{\mathfrak{g}}$ by setting $\hat{\varphi}(r_0 + r_1 + s) = \varphi_0(r_1 + s) + \varphi_1(r_0)$. Indeed, according to the previous results we have

$$[\varphi_0(r_1 + s) + \varphi_1(r_0), r_0' + r_1' + s'] = [\varphi_1(r_0), r_0']$$

and

$$[\varphi_0(r_1' + s') + \varphi_1(r_0'), r_0 + r_1 + s] = [\varphi_1(r_0'), r_0]$$

where (r_0, r_1, s) and (r_0', r_1', s') are elements of $\mathfrak{r}^{\mathfrak{s}} \times [\mathfrak{r}, \mathfrak{s}] \times \mathfrak{s} \simeq \mathfrak{g}$. Since φ_1 is an element of $h_{\mathfrak{r}^{\mathfrak{s}}}$ we have

$$[\hat{\varphi}(X), Y] = [\hat{\varphi}(Y), X]$$

for any pair (X, Y) in $\mathfrak{g} \times \mathfrak{g}$. That proves the inclusion

$$\operatorname{Hom}_K([\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}, Z(\mathfrak{g})) \oplus h_{\mathfrak{r}^{\mathfrak{s}}} \cap \operatorname{Hom}_K\big(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})\big) \subset h_{\mathfrak{g}}$$

which ends the proof of Theorem II.

Corollary II.1. *For a Lie algebra $\mathfrak{g} \simeq \mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$, the space $h_{\mathfrak{g}}$ is zero if and only if the ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is zero.*

Proof. The sufficient condition is trivial. Conversely, let us suppose that $h_{\mathfrak{g}}$ is zero. As we did before, we may suppose that the ground field K is algebraically closed. If $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ were different from zero, by applying a classical Lie theorem to the solvable Lie algebra $\mathfrak{r}$, one could find a nonzero element v_0 in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ and a linear form $\lambda \in \mathfrak{r}^*$ such that for any $X \in \mathfrak{r}$ one gets

$$[X, v_0] = \lambda(X)v_0.$$

Since $h_{\mathfrak{g}}$ is zero, so is $Z(\mathfrak{g})$, so that the linear form λ is different from zero. Let us define the linear map φ of $\mathfrak{g}$ into itself by putting

$$\varphi(r + s) = \lambda(r)v_0$$

for all $(r, s) \in \mathfrak{r} \times \mathfrak{s}$. Thus, given (r, s) and (r', s') in $\mathfrak{r} \times \mathfrak{s}$ we have

$$[\varphi(r + s), r' + s'] = [\lambda(r)v_0, r' + s'] = \lambda(r)[v_0, r'] = -\lambda(r)\lambda(r')v_0$$

and

$$[\varphi(r' + s'), r + s] = [\lambda(r')v_0, r + s] = \lambda(r')[v_0, r] = -\lambda(r')\lambda(r)v_0.$$

We must conclude that the linear map φ is a nonzero element of $h_{\mathfrak{g}}$, which is contrary to our assumption. Corollary II.1 is proved.

Corollary II.2. *Let $\mathfrak{g}$ be a Lie algebra with nilpotent radical $\mathfrak{r}$. Then if the center $Z(\mathfrak{g})$ is zero so is the Lie algebra $h_{\mathfrak{g}}$.*

Proof. By Corollary II.1, if $h_{\mathfrak{g}}$ were not zero, the same would hold for the ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$. Applying the theorem of Engel, one would have a nonzero element X_0 in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) \cap Z(\mathfrak{r})$. Such an element X_0 would lie in the center $Z(\mathfrak{g})$.

Keeping in mind our geometrical interest in the prolongation $\mathrm{ad}(\mathfrak{g})^{(1)}$, the previous results lead to this result.

THEOREM III. *Let $\mathfrak{g}$ be a Lie algebra with a decomposition $\mathfrak{g} \simeq \mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}$. The first prolongation $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ of the linear space $\mathrm{ad}(\mathfrak{g})$ is isomorphic to the factor space $h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}))/h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}_K(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{g}))$.*

The proof is an immediate consequence of Theorem II together with the exact sequence $0 \to \mathrm{Hom}(\mathfrak{g}, Z(\mathfrak{g})) \to h_{\mathfrak{g}} \to (\mathrm{ad}(\mathfrak{g}))^{(1)} \to 0$.

COROLLARY III.1. *Let $\mathfrak{g}$ be a Lie algebra with a Levi decomposition $\mathfrak{r} \oplus \mathfrak{s}$. If $\mathfrak{r}^{\mathfrak{s}}$ is commutative then $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ is zero.*

PROOF. We already proved that the ideal $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is equal to $\mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$. Since $\mathfrak{r}^{\mathfrak{s}}$ is commutative we get $h_{\mathfrak{r}^{\mathfrak{s}}} = \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, \mathfrak{r}^{\mathfrak{s}})$ and $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{g}^{\mathfrak{r}^{\mathfrak{s}}} = \mathfrak{g}^{\mathfrak{g}} = Z(\mathfrak{g})$. Therefore, we have $h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{g})) = \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{g}))$.

COROLLARY III.2. *Let $\mathfrak{g}$ be a Lie algebra such that some $\mathfrak{r}^{\mathfrak{s}}$ is an ideal in $\mathfrak{g}$. Then $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ is isomorphic to $(\mathrm{ad}(\mathfrak{r}^{\mathfrak{s}}))^{(1)}$.*

PROOF. Since $\mathfrak{r}^{\mathfrak{s}}$ is an ideal of $\mathfrak{g}$ we have $\mathfrak{r}^{\mathfrak{s}} = D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}) = \mathfrak{r}^{\mathfrak{s}} \cap \mathfrak{r}^{[\mathfrak{r},\mathfrak{s}]}$. On the other hand, we can write $[\mathfrak{r}^{\mathfrak{s}}]^{\mathfrak{r}^{\mathfrak{s}}} = Z(\mathfrak{r}^{\mathfrak{s}})$ so that

$$Z(\mathfrak{r}^{\mathfrak{s}}) \subset [\mathfrak{r}^{\mathfrak{s}} \oplus [\mathfrak{r}, \mathfrak{s}] \oplus \mathfrak{s}]^{\mathfrak{r}^{\mathfrak{s}}} = Z(\mathfrak{g}).$$

That proves the equality $Z(\mathfrak{r}^{\mathfrak{s}}) = Z(\mathfrak{g})$. We apply Theorem III and we obtain $(\mathrm{ad}(\mathfrak{g}))^{(1)} \simeq h_{\mathfrak{r}^{\mathfrak{s}}}/\mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, Z(\mathfrak{r}^{\mathfrak{s}})) \simeq (\mathrm{ad}(\mathfrak{r}^{\mathfrak{s}}))^{(1)}$.

PROPOSITION 2.1. *Let $\mathfrak{g}$ be a Lie algebra. Then for any Levi subalgebra $\mathfrak{s}$, the subspace $h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}}, D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}}))$ is an ideal of $h_{\mathfrak{r}^{\mathfrak{s}}}$. Furthermore, the subspace $h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}}) \cap D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ is an ideal of $\mathfrak{g}$ which does not depend on the choice of $\mathfrak{s}$.*

PROOF. Let $\varphi \in h_{\mathfrak{r}^{\mathfrak{s}}}$ and let $\psi \in h_{\mathfrak{r}^{\mathfrak{s}}} \cap \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}})$ so that $[\varphi, \psi]$ lies in $h_{\mathfrak{r}^{\mathfrak{s}}}$. It remains to prove that for all (r_0, r, s) in $\mathfrak{r}^{\mathfrak{s}} \times \mathfrak{r} \times \mathfrak{s}$ we have $[[\varphi, \psi](r_0), [r, s]] = 0$. Here

$$\begin{aligned}[][[\varphi, \psi](r_0), [r, s]] &= [\varphi\psi(r_0) - \psi\varphi(r_0), [r, s]] \\ &= [\varphi\psi(r_0), [r, s]] = [[\varphi\psi(r_0), r], s].\end{aligned}$$

If $\mathfrak{i}$ is an ideal of a Lie algebra $\mathfrak{g}$ and $\varphi \in h_{\mathfrak{g}}$, for all $v \in \mathfrak{g}$, we get $[\varphi(\mathfrak{i}), v] = [\varphi(v), \mathfrak{i}] \subset \mathfrak{i}$ so that in the previous case $[\varphi\psi(r_0), r]$ lies in $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$ and the first statement holds. Now let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two Levi subalgebras of $\mathfrak{g}$. A theorem of Malcev and Harish-Chandra tells us that there is an element X_0 of the nilpotent radical of $\mathfrak{g}$ such that $\mathfrak{s}_2 = e^{\mathrm{ad}(X_0)}(\mathfrak{s}_1)$. Since $e^{\mathrm{ad}(X_0)}$ preserve every ideal of $\mathfrak{g}$ we have $e^{\mathrm{ad}(X_0)}(D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1})) = D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1})$. Since $\mathfrak{r}^{\mathfrak{s}_1}$ and $\mathfrak{r}^{\mathfrak{s}_2}$ must be conjugated by $e^{\mathrm{ad}(X_0)}$, so must the ideals $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1})$ and $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_2})$, and one concludes that $D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_1}) = D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}_2})$.

Let us illustrate the main results by a few examples.

EXAMPLE 1. Let $\mathfrak{g}$ be a semisimple Lie algebra. The radical $\mathfrak{r}$ of $\mathfrak{g}$ being zero, Theorem I gives $h_{\mathfrak{g}} = \{0\}$, so that we get $(\mathrm{ad}(\mathfrak{g}))^{(1)} = \{0\}$.

EXAMPLE 2. If $\mathfrak{g}$ is a reductive Lie algebra then we get $Z(\mathfrak{g}) = \mathfrak{r}^{\mathfrak{s}} = \mathfrak{r}$. By the Corollary III.1, we have $(\mathrm{ad}(\mathfrak{g}))^{(1)} = \{0\}$.

EXAMPLE 3. Let $\mathfrak{g}$ be the affine Lie algebra $\mathbf{R}^2 \times sl(2, \mathbf{R})$ and let (u, X) be an element of $\mathfrak{g}$. Then we get $\mathrm{ad}(u, X) = [\begin{smallmatrix} X & -\delta u \\ 0 & \mathrm{ad}\, X \end{smallmatrix}]$ where $\delta u(Y) = Yu$ for $Y \in sl(2, \mathbf{R})$. Since $sl(2, \mathbf{R})$ is irreducible on $\mathbf{R}^2$, we have $\mathfrak{r}^{sl(2,\mathbf{R})} = \{0\}$, and Theorem III gives $(\mathrm{ad}(\mathfrak{g}))^{(1)} = \{0\}$.

EXAMPLE 4. Let $\mathfrak{g}$ be the Lie algebra $\mathbf{R}^5 \# sl(2, \mathbf{R})$, with the bracket given by

$$[((a, b, c, \alpha, \beta), X), ((a', b', c', \alpha', \beta'), X')] \\ = (bc' - b'c + \alpha\beta' - \alpha'\beta, 0, 0, X(\alpha', \beta') - X'(\alpha, \beta), [X, X']).$$

Let us take $\mathfrak{s}$ to be the subalgebra $\{0, 0, 0, 0, 0\} \# sl(2, \mathbf{R})$. It is clear that

$$\mathfrak{r}^{\mathfrak{s}} = \mathbf{R}^3 \times \{(0,0)\} \# \{0\},$$
$$[\mathfrak{r}, \mathfrak{s}] = \{(0,0,0)\} \times \mathbf{R}^2 \# \{0\},$$
$$Z(\mathfrak{g}) = Z(\mathfrak{r}^{\mathfrak{s}}) = \mathbf{R} \times \{0,0,0,0\} \# \{0\}.$$

Since $\mathfrak{r}^{\mathfrak{s}}$ is an ideal in $\mathfrak{g}$, by Corollary III.2, $(\mathrm{ad}(\mathfrak{g}))^{(1)}$ is isomorphic to the first prolongation of the inner derivations of the Heisenberg algebra $\mathfrak{r}^{\mathfrak{s}}$, which is the set of those $S \in \mathrm{Hom}(\mathfrak{r}^{\mathfrak{s}} \times \mathfrak{r}^{\mathfrak{s}}, \mathfrak{r}^{\mathfrak{s}})$ defined by

$$S((a, b, c), (a', b', c')) = ((\lambda b + \mu c)b' + (\mu b + \nu c)(c', 0, 0)$$

where $(\lambda, \mu, \nu) \in \mathbf{R}^3$.

3. Return to differential geometry. We begin by explaining the geometric interest of the ideal $\mathcal{I} = h_{\mathfrak{r}^{\mathfrak{s}}}(\mathfrak{r}^{\mathfrak{s}}) \cap D_{\mathfrak{g}}^{\infty}(\mathfrak{r}^{\mathfrak{s}})$. One easily verifies that $\mathcal{I}$ is the minimal ideal of $\mathrm{ad}(\mathfrak{g})$ such that the first prolongation of $\mathrm{ad}(\mathfrak{g})$ coincides with that of $\mathcal{I}$, so that

$$(\mathrm{ad}(\mathfrak{g}))^{(1)} = \mathcal{I}^{(1)}.$$

This gives another understanding of Proposition 2.1. Moreover, the geometrical statement $(\mathfrak{R}_1)$ is a direct consequence of the above remark. The geometrical statements $(\mathfrak{R}_2)$, $(\mathfrak{R}_3)$ and $(\mathfrak{R}_4)$ are consequences of Corollaries I.1, II.2 and III.2, respectively.

Take a left invariant torsion free connection ∇ on a Lie group $\mathcal{G}$ and assume that its holonomy group is a subgroup of $\mathrm{Int}(\mathfrak{g})$. One observes that the space $\mathrm{ad}(\mathfrak{g})^{(1)}$ provides a parametrization of the set of all left invariant torsion free connections which are adapted to the $\mathrm{Int}(\mathfrak{g})$-structure obtained from the holonomy bundle of ∇ (see §1).

Our last remark applies to the case of solvable Lie groups which cannot be handled by the techniques used in this work. We may observe that for such a Lie group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$ the linear Lie algebra $h_{\mathfrak{g}}$ is always different from zero. Let $\mathfrak{g}$ be a solvable Lie algebra. If $Z(\mathfrak{g}) \neq 0$, $\mathrm{Hom}_K(\mathfrak{g}, Z(\mathfrak{g}))$ is included in $h_{\mathfrak{g}}$. If $Z(\mathfrak{g}) = 0$ then any ξ in $Z([\mathfrak{g}, \mathfrak{g}]) - \{0\}$ gives us a nonzero element ad_{ξ} in $h_{\mathfrak{g}}$. Thus, for any solvable Lie algebra with $Z(\mathfrak{g}) = 0$ the first prolongation $\mathrm{ad}(\mathfrak{g})^{(1)}$ is never zero.

ADDENDUM. The authors are indebted to the referee for many useful suggestions which led to better formulation of some theorems.

Bibliography

1. V. Guillemin, *A Jordan Hölder decomposition for a certain class of infinite dimensional Lie algebras*, J. Differential Geom. **2** (1968).

2. N. Jacobson, *Lie algebras*, Tracts in Math., no. 10, Interscience, New York, 1962.

3. J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 293–329.

4. I. Singer and S. Sternberg, *The infinite groups of Lie and Cartan*, J. Analyse Math. **15** (1965), 1–114.

Université des Sciences et Techniques du Languedoc, Place Eugène Baraillon, 34060-Montpellier Cedex, France